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Generating functions for plethysms of finite and continuous groups†

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Abstract. Generating functions (also Molien functions or Poincaré series) formerly used to study only completely symmetrical components of tensor products (symmetrical plethysms) are developed for calculation of general plethysms. Particular generating functions involving compact Lie groups as well as finite groups are found and corresponding integrity bases are calculated. The first examples of a novel type of generating function are found. They correspond to antisymmetric or symmetric plethysms in which the number of factors in the tensor product is fixed but the representation on which the plethysms are based runs through all irreducible representations of SU(2). A new type of transitive relation between pairs of Lie algebra, the subjoining of one to another, is demonstrated and exploited.

1. Introduction

One of the chief quantum principles, the impossibility of distinguishing identical particles, imposes a definite symmetry with respect to interchange of particles on the wavefunction of a many-particle system; the wavefunction is symmetric (antisymmetric) with respect to interchange of bosons (fermions). For a system of p identical particles with internal degrees of freedom, the spatial and internal parts of the wavefunction transform according to two irreducible representations of the symmetric group S_p, which complement each other to yield this required overall symmetry. The spatial states of a single particle usually constitute a Γ multiplet or Γ tensor (which transforms according to a definite representation Γ) of a finite or compact Lie group G; similarly, the internal states form a Γ' tensor of a group G′.

Thus, in constructing a p-particle wavefunction, one encounters the following mathematical problem: given an irreducible representation $\{\lambda\}$ of S_p and a Γ tensor of G, decompose $\Gamma^{\{\lambda\}}$, the component of the tensor product of p copies of the Γ tensor with exchange symmetry $\{\lambda\}$, into a direct sum of irreducible tensors of G. This problem is called the calculation of the plethysm $\Gamma^{\{\lambda\}}$. One writes

$$\Gamma\otimes\Gamma\otimes\ldots\Gamma=\bigoplus_{\{\lambda\}}\Gamma^{\{\lambda\}}\{\lambda\};\qquad \Gamma^{\{\lambda\}}=\bigoplus_{j}\Gamma^{j}m_j. \tag{1.1}$$

Here m_j is the number of irreducible Γ_j tensors of G in the plethysm $\Gamma^{\{\lambda\}}$. We specify irreducible representations of S_p by $\{\lambda\}=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ where λ_i is the number of

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columns of i boxes in the corresponding Young tableau (see figure 1); n is the dimension of the Γ tensor and

$$p = \sum_{i=1}^{n} i\lambda_i. \tag{1.2}$$

The possibility of the decomposition (1.1) follows from the fact that S_p transformations commute with G transformations, so that the tensor product $\Gamma\otimes\Gamma\otimes\ldots\otimes\Gamma$ (p times) can be decomposed according to the group $G\times S_p$.

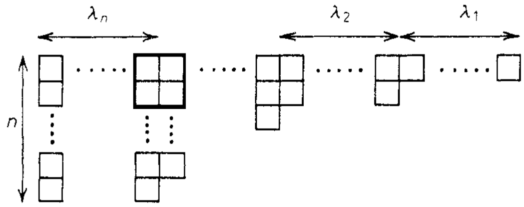

Figure 1. The Young tableau corresponding to the representation $\{\lambda_1\lambda_2\ldots\lambda_n\}$ of S_p.

Of equal importance to the plethysm computation is the explicit construction of the corresponding tensors. It turns out that for given G and Γ, they may be constructed from a small number of 'elementary' tensors which constitute the integrity basis of the problem.

Our purpose in this paper is twofold: to calculate plethysms for finite and compact Lie groups, and to study the corresponding integrity bases. Both problems are solved at the same time by the generating function technique.

The concept of plethysm was introduced by Littlewood (1943a, b, 1958). Its importance in quantum physics has been stressed by a number of authors (Butler and Wybourne 1971, Moshinsky 1967, Kota 1977).

Several recent papers (Gaskell *et al* 1978, Patera *et al* 1978, Desmier and Sharp 1979, Patera and Sharp 1979a, b) derive generating functions for polynomial tensors, i.e., tensors whose components are polynomials in the components of a given Γ tensor of a finite or compact Lie group G. Polynomial tensors of degree p constitute the symmetric plethysm corresponding to the representation $\{\lambda\}=\{p, 0, \ldots, 0\}$ of S_p; the Young tableau consists of one row of p boxes. In this paper we generalise these methods to include plethysms of all symmetries.

2. Computation of plethysm generating functions

In this section we first elaborate the well known relation between irreducible representations of the symmetric group S_p and those of the group $U(n)$ of unitary $n\times n$ matrices (Weyl 1946). One is led naturally to the conclusion that the calculation of plethysms based on a Γ tensor of a group G is equivalent to calculation of branching rules for $U(n)\supset G$, where n is the dimension of Γ.

To denote a representation of G, or a tensor which transforms by it, we use Γ, Γ_i, etc; if G is a compact Lie group and the tensor is irreducible we may use $(\mu)=(\mu_1, \mu_2, \ldots, \mu_l)$, where μ_i are non-negative integers and l is the rank of G. Irreducible representations of S_p are labelled by $\{\lambda\}=\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ as explained in § 1. We specify irreducible representations of U(n) by $(\lambda)=(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i are non-negative integers.

The tensor product $(1, 0, \ldots, 0)^p$ of p copies of the defining representation of U(n) is a (reducible) tensor of the product group $\mathrm{U}(n)\times \mathrm{S}_p$. It decomposes as

$$(1, 0, \ldots, 0)^p = \bigoplus_{\{\lambda\}} (\lambda)\{\lambda\} \tag{2.1}$$

where the sum is over those irreducible representations of S_p whose Young tableaux contain no columns longer than n. The plethysm $(1, 0, \ldots, 0)^{\{\lambda\}}$ is the coefficient of $\{\lambda\}$ on the right of (2.1), i.e., an irreducible (λ) tensor of U(n). The multiplicity of $(\lambda)\{\lambda\}$ in (2.1) is unity, while the multiplicity of (λ) of U(n) is the dimension $D_{\{\lambda\}}$ of the S_p representation $\{\lambda\}$:

$$D_{\{\lambda\}} = p! \prod_{i=1}^{p-1} \prod_{j=i+1}^{p} (l_i - l_j) \Big/ \prod_{i=1}^{p} l_i! \tag{2.2}$$

where

$$l_i = \sum_{k=i}^{p} \lambda_k + p - i \qquad i = 1, 2, \ldots, p. \tag{2.3}$$

All $D_{\{\lambda\}}$ (λ) tensors are obtained from any one by applying the permutations of S_p. We ignore this obvious repetition of U(n) tensors from now on.

These considerations imply the following procedure for finding plethysms based on a Γ tensor of a group G (King 1974): embed G in U(n), where n is the dimension of Γ, so $(1, 0, \ldots, 0)$ of U(n) contains Γ once. Then the plethysm $\Gamma^{\{\lambda\}}$ consists of the tensors $\oplus_j \Gamma_j$ of G into which a (λ) tensor of U(n) decomposes. Our problem is reduced to that of finding the generating function for the branching rules $\mathrm{U}(n) \supset \mathrm{G}$. General methods have been described previously (Gaskell *et al* 1978, Patera and Sharp 1979a). In fact, since $\mathrm{U}(n) = \mathrm{SU}(n) \times \mathrm{U}(1)$ it is necessary to compute only the generating function for $\mathrm{SU}(n) \supset \mathrm{G}$; the U(1) label λ_n is the number of columns of n boxes, and its value has no effect on the G content of the plethysm.

Incidentally, a useful dimensionality relation follows from (2.1):

$$n^p = \sum_{\{\lambda\}} D_{\{\lambda\}} D_{(\lambda)}. \tag{2.4}$$

The dimension $D_{(\lambda)}$ of the U(n) representation (λ) is

$$D_{(\lambda)} = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (l_i - l_j) \Big/ \prod_{i=1}^{n-1} i! \tag{2.5}$$

where

$$l_i = \sum_{k=i}^{n} \lambda_k + n - i. \tag{2.6}$$

By now, generating functions for $\mathrm{SU}(n) \supset \mathrm{G}$, with G semi-simple and $n \leqslant 4$, are known for most cases of interest. The *ab initio* computations are generally lengthy when $n = 4$, and for some G are about the limit of what can be computed by hand. Since plethysms involving U(n), $n \geqslant 5$, are also interesting, it is important to use any available shortcuts. In the remainder of this section we list a few ways of obtaining new generating functions by combination of known generating functions or by making use of known integrity bases:

(1) Often one needs plethysms based on a reducible Γ tensor of G; the generating function can be constructed in terms of those based on the irreducible tensors comprising Γ. For simplicity suppose $\Gamma = \Gamma_1 \oplus \Gamma_2$, where the irreducible tensors Γ_1 and Γ_2 are of dimension n_1 and n_2 respectively, and $n_1 + n_2 = n$. First one writes the generating function for $SU(n) \supset SU(n_1) \times SU(n_2) \times U(1)$ (it is possible to construct such a generating function for any n_1, n_2 using the results of Mickelsson (1970)). Then one 'substitutes' into it the generating functions for $SU(n_1) \supset G$ and $SU(n_2) \supset G$. What is involved in such a 'substitution' is explained in §§ 3 and 4. At this stage the G representations occur as tensor products of pairs of representations, one from $SU(n_1)$ and one from $SU(n_2)$. To decompose the products, one needs the Clebsch–Gordan generating function if G is continuous, or the Clebsch–Gordan series if G is finite. Examples of the procedure are found in §§ 3 and 4.

(2) It frequently happens that G is not maximal in $SU(n)$, but occurs in the chain $U(n) \supset SU(n) \supset G' \supset G$. The problem may then be solved in two steps. One starts with the generating functions for $U(n) \supset G'$ and for $G' \supset G$ and then 'substitutes' the second in the first. Moreover it is easy to retain the information concerning G' in the $U(n) \supset G$ generating function.

(3) Even when G is maximal in $SU(n)$ it is sometimes possible to find a group G'' which plays the same role as the intermediate group G' of the preceding paragraph. An example is found is § 3 and is discussed in § 6.

(4) Elements of integrity bases and their syzygies (polynomial relations between elements of a basis) are found in the literature for many group–subgroup decompositions; it is straightforward to rewrite such information in terms of a generating function.

(5) Sometimes an integrity basis and its syzygies can be guessed after examining a number of low-lying representations of the parent group. The corresponding generating function can be checked by converting it into the generating function for representations of a lower subgroup or for weights and comparing it with the known generating function. When the conversion is too lengthy to carry out analytically, the comparison can be made numerically with the help of a computer by giving random values to the dummy variables on which the generating function depends. Although inelegant mathematically, this method is practical when used with caution.

3. Plethysms for compact Lie groups

In this section we describe the generating function and related notation, explain the procedure for substituting one generating function into another, and calculate a number of plethysms for compact Lie groups.

The generating functions depends on $n + l$ auxiliary variables; n is the dimension of Γ and l is the rank of G. When expanded into a power series,

$$\mathscr{F}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n; M_1, M_2, \ldots, M_l) = \sum_{\lambda} \Lambda^{\lambda} \sum_{\mu} m_{\lambda,\mu} M^{\mu}$$
$$\Lambda^{\lambda} \equiv \prod_{i=1}^{n} \Lambda_i^{\lambda_i} \qquad M^{\mu} \equiv \prod_{j=1}^{l} M_j^{\mu_j}, \tag{3.1}$$

the first n variables carry the plethysm labels $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the other l variables carry the G-representation labels $\mu_1, \mu_2, \ldots, \mu_l$. A term $\Lambda^{\lambda} m_{\lambda,\mu} M^{\mu}$ in (3.1) says that the irreducible representation (μ) of G is contained in the plethysm $\Gamma^{\{\lambda\}}$ exactly $m_{\lambda,\mu}$ times.

We now explain how one generating function may be substituted into another. Let $\mathscr{H}(N, M)$ be the generating function for $G' \supset G$ (G-representation labels carried by M, G′ labels by N) and $\mathscr{J}(\Lambda, N)$ be that for $G'' \supset G'$ (G″ labels carried by Λ). Then the generating function $\mathscr{F}(\Lambda, N, M)$ for $G'' \supset G' \supset G$ is

$$\begin{aligned}\mathscr{F}(\Lambda, N, M) &= \sum \mathrm{Res}_{N'}\left[\mathscr{J}(\Lambda, N(N')^{-1})\mathscr{H}(N', M) \prod_{i=1}^{l_{G'}} (N'_i)^{-1}\right] \\ &= \sum \mathrm{Res}_{N'}\left[\mathscr{J}(\Lambda, N')\mathscr{H}(N(N')^{-1}, M) \prod_{i=1}^{l_{G'}} (N'_i)^{-1}\right]\end{aligned} \tag{3.2}$$

where $\Sigma\, \mathrm{Res}_{N'}$ means the sum of residues of poles of the variables N' inside their unit circles. In order to decide whether a pole is inside the unit circle, regard variables other than the variables N' as being less than unity in magnitude, and the N' themselves as being just outside the unit circle. The two forms (3.2) are obtainable from each other by the substitution $N'_i \to (N'_i)^{-1}$, $i = 1, 2, \ldots, l_{G'}$; use whichever is easier to evaluate. The role of the variables N in (3.2) is to preserve the information about the intermediate group G′; if that information is not required, the variables N may be set equal to unity.

We turn now to the generating function for plethysms based on a reducible tensor $\Gamma' \oplus \Gamma''$ of G. Suppose that the generating functions $\mathscr{E}'(\Lambda', M')$ and $\mathscr{E}''(\Lambda'', M'')$ for $SU(n') \supset G$ and $SU(n'') \supset G$, where n' and n'' are the dimensions of Γ' and Γ'' respectively, are known. We also assume that we have the generating function $\mathscr{H}(\Lambda, \Lambda', \Lambda'', Z)$ for $SU(n'+n'') \supset SU(n') \times SU(n'') \times U(1)$. The U(1) label, carried by Z, is chosen to be the degree in the tensor Γ'.

In order to get the function $\mathscr{E}(\Lambda, M, Z)$ for $SU(n'+n'') \supset G \times U(1)$ we proceed as follows. Substituting $\mathscr{E}'(\Lambda', M')\mathscr{E}''(\Lambda'', M'')$ (it is the generating function for $SU(n') \times SU(n'') \supset G \times G$) into $\mathscr{H}(\Lambda, \Lambda', \Lambda'', Z)$ according to equation (3.2), one obtains a generating function $\mathscr{J}(\Lambda, M', M'', Z)$ for $SU(n'+n'') \supset G \times G \times U(1)$. For the final step one needs the generating function $\mathscr{C}(M', M'', M)$ for $G \times G \supset G$, i.e., the Clebsch–Gordan generating function for the group G. Substituting $\mathscr{C}(M', M'', M)$ into $\mathscr{J}(\Lambda, M', M'', Z)$ according to (3.2) yields $\mathscr{E}(\Lambda, M, Z)$. Multiplying $\mathscr{E}$ by $(1-\Lambda_{n'+n''}Z^{n'})^{-1}$, one obtains the generating function $\mathscr{F}(\Lambda, M, Z)$ for plethysms based on $\Gamma' \oplus \Gamma''$ (the dummies Λ now include the additional $U(n'+n'')$ dummy $\Lambda_{n'+n''}$; the variable Z carries the degree in Γ').

In case G is the special unitary group $SU(n)$, and Γ' and/or Γ'' its defining representation, the functions $\mathscr{F}'$ and/or $\mathscr{F}''$ are trivial and may be substituted into $\mathscr{H}$, to obtain $\mathscr{J}$, by the replacements $\Lambda' \to M'$ and/or $\Lambda'' \to M''$.

There follows a number of examples of generating functions for plethysms.

3.1. Plethysms based on (1, 0, . . . , 0) of U(n)

For this trivial case it is obvious from the discussion of § 2 that the required generating function is

$$\mathscr{F}(\Lambda; M) = \left(\prod_{i=1}^{n} (1 - \Lambda_i M_i)\right)^{-1}. \tag{3.3}$$

(3.3) becomes the generating function for plethysms based on $(1, 0, \ldots, 0)$ of $SU(n)$ if M_n is set equal to unity.

3.2. Plethysms based on (2) of O(3)

Here (2) is the three-dimensional (vector) representation. The O(3) content of U(3) representations is well known (Bargmann and Moshinsky 1961) and the required generating function is

$$\mathscr{F}(\Lambda_1, \Lambda_2, \Lambda_3; M) = \frac{1+\Lambda_1\Lambda_2 M^2}{(1-\Lambda_1^2)(1-\Lambda_1 M^2)(1-\Lambda_2^2)(1-\Lambda_2 M^2)(1-\Lambda_3)}. \quad (3.4)$$

The exponents of M labelling the O(3) representations are double the angular momenta.

3.3. Plethysms based on (10) of Sp(4)

(10) is the quartet of Sp(4). Since Sp(4) and SU(4) are locally isomorphic to O(5) and O(6) respectively, the generating function for SU(4) ⊃ Sp(4) is the same as that for O(6) ⊃ O(5). The well known integrity basis for that problem (e.g., Sharp and Lam 1969) allows one to write

$$\mathscr{F} = [(1-\Lambda_1 M_1)(1-\Lambda_2 M_2)(1-\Lambda_2)(1-\Lambda_3 M_1)(1-\Lambda_1\Lambda_3 M_2)(1-\Lambda_4)]^{-1}. \quad (3.5)$$

The factor $(1-\Lambda_4)^{-1}$ takes account of the fact that we are dealing here with U(4).

3.4. Plethysms based on (1)(1) of SU(2) × SU(2)

The embedding of SU(2) × SU(2) in SU(4) is that of the Wigner supermultiplet model of nuclear physics. Our SU(2) representation labels are all integers; they equal twice the corresponding angular momenta or isospins. The integrity basis for SU(4) ⊃ SU(2) × SU(2) is known (Sharp and Lam 1969, Patera and Sharp 1979a); when transcribed as a generating function for plethysms it becomes

$$\begin{aligned}\mathscr{F} = {} & \frac{1}{(1-\Lambda_1^2)(1-\Lambda_3^2)(1-\Lambda_4)(1-\Lambda_2 M_1^2)(1-\Lambda_2 M_2^2)} \\ & \times\left(\frac{1}{(1-\Lambda_1 M_1 M_2)(1-\Lambda_3 M_1 M_2)(1-\Lambda_1\Lambda_3 M_1^2)}\right. \\ & + \frac{\Lambda_1\Lambda_3 M_2^2}{(1-\Lambda_1 M_1 M_2)(1-\Lambda_3 M_1 M_2)(1-\Lambda_1\Lambda_3 M_2^2)} \\ & + \frac{\Lambda_2^2+\Lambda_1\Lambda_2\Lambda_3 M_1^2}{(1-\Lambda_1 M_1 M_2)(1-\Lambda_2^2)(1-\Lambda_1\Lambda_3 M_1^2)} \\ & + \frac{\Lambda_1\Lambda_2^2\Lambda_3 M_2^2+\Lambda_1\Lambda_2 M_1 M_2}{(1-\Lambda_2^2)(1-\Lambda_1 M_1 M_2)(1-\Lambda_1\Lambda_3 M_2^2)} \\ & + \frac{\Lambda_2^2\Lambda_3 M_1 M_2+\Lambda_2\Lambda_3 M_1 M_2}{(1-\Lambda_2^2)(1-\Lambda_3 M_1 M_2)(1-\Lambda_1\Lambda_3 M_1^2)} \\ & \left. + \frac{\Lambda_1\Lambda_2^2\Lambda_3^2 M_1 M_2^3+\Lambda_1\Lambda_2\Lambda_3 M_2^2}{(1-\Lambda_2^2)(1-\Lambda_3 M_1 M_2)(1-\Lambda_1\Lambda_3 M_2^2)}\right). \end{aligned} \quad (3.6)$$

3.5. Plethysms based on (01) of O(5)

No integrity basis or generating function for SU(5) ⊃ O(5) appears in the literature; it would be extremely tedious to deduce it from the generating function for SU(5) weights. O(5) is maximal in SU(5), but it still possible to utilise an intermediate subgroup, SU(4), to facilitate the calculation. Some multiplets of O(5) appear in SU(4) multiplets with negative coefficients. Details are found in § 6. The required plethysm generating function turns out to be

$$
\begin{aligned}
\mathscr{F} = {} & \frac{1}{(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3^2)(1-\Lambda_4^2)(1-\Lambda_5)(1-\Lambda_2 M_1^2)(1-\Lambda_3 M_1^2)(1-\Lambda_1 M_2)(1-\Lambda_4 M_2)} \\
& \times \Bigg(\frac{1+\Lambda_1\Lambda_2\Lambda_3 M_1^2+\Lambda_1\Lambda_3\Lambda_4 M_1^2+\Lambda_1\Lambda_2\Lambda_3\Lambda_4 M_1^2}{(1-\Lambda_1\Lambda_3 M_1^2)(1-\Lambda_1\Lambda_4 M_1^2)} \\
& + \frac{\Lambda_2\Lambda_4 M_1^2+\Lambda_2\Lambda_3\Lambda_4 M_1^2+\Lambda_1\Lambda_2\Lambda_4 M_1^2+\Lambda_1\Lambda_2^2\Lambda_3\Lambda_4^2 M_1^2}{(1-\Lambda_2\Lambda_4 M_1^2)(1-\Lambda_1\Lambda_4 M_1^2)} \\
& + \frac{\begin{array}{l}\Lambda_3^2 M_2^2+\Lambda_3\Lambda_4 M_2+\Lambda_1\Lambda_2\Lambda_3^2 M_1^2 M_2+\Lambda_1\Lambda_2\Lambda_3^2 M_1^2 M_2^2 \\ \quad +\Lambda_1\Lambda_3^2 M_1^2 M_2+\Lambda_2\Lambda_3^2\Lambda_4 M_2^2+\Lambda_1\Lambda_2\Lambda_3^2\Lambda_4 M_1^2 M_2+\Lambda_1\Lambda_3^3\Lambda_4 M_1^2 M_2^2\end{array}}{(1-\Lambda_1\Lambda_3 M_1^2)(1-\Lambda_3^2 M_2^2)} \\
& + \frac{\begin{array}{l}\Lambda_2^2 M_2^2+\Lambda_1\Lambda_2 M_2+\Lambda_2^2\Lambda_3\Lambda_4 M_1^2 M_2+\Lambda_2^3\Lambda_3\Lambda_4 M_1^2 M_2^2 \\ \quad +\Lambda_2^2\Lambda_4 M_1^2 M_2+\Lambda_1\Lambda_2^2\Lambda_3 M_2^2+\Lambda_1\Lambda_2^2\Lambda_3\Lambda_4 M_1^2 M_2+\Lambda_1\Lambda_2^2\Lambda_4 M_1^2 M_2^2\end{array}}{(1-\Lambda_2\Lambda_4 M_1^2)(1-\Lambda_2^2 M_2^2)} \\
& + \frac{1}{(1-\Lambda_2^2 M_2^2)(1-\Lambda_3^2 M_2^2)}(\Lambda_2^2\Lambda_3^2 M_2^4+\Lambda_1\Lambda_2\Lambda_3^2 M_2^3+\Lambda_2^2\Lambda_3\Lambda_4 M_2^3 \\
& +\Lambda_2\Lambda_3 M_2+\Lambda_2\Lambda_3 M_1^2 M_2+\Lambda_1\Lambda_2\Lambda_3\Lambda_4 M_2^2+\Lambda_1\Lambda_2^2\Lambda_3^3 M_2^4+\Lambda_2^3\Lambda_3^2\Lambda_4 M_2^4 \\
& +\Lambda_1\Lambda_2^2\Lambda_3 M_1^2 M_2^2+\Lambda_2\Lambda_3^2\Lambda_4 M_1^2 M_2^2+\Lambda_2^2\Lambda_3^2 M_1^2 M_2^2+\Lambda_1\Lambda_2^2\Lambda_3^2\Lambda_4 M_2^4 \\
& +\Lambda_1\Lambda_2^2\Lambda_3^2\Lambda_4 M_1^2 M_2^3+\Lambda_1\Lambda_2^3\Lambda_3^2 M_1^2 M_2^3+\Lambda_2^2\Lambda_3^3\Lambda_4 M_1^2 M_2^3 \\
& +\Lambda_1\Lambda_2^3\Lambda_3^3\Lambda_4 M_1^2 M_2^4)\Bigg). \qquad (3.7)
\end{aligned}
$$

3.6. Plethysms based on (1, 0, . . . , 0) ⊕ (0, 0, . . . , 0) of SU(n)

The embedding of SU(n) in SU($n+1$) is the canonical one, whose integrity basis is well known. The corresponding plethysm generating function is

$$
\mathscr{F} = \left((1-\Lambda_{n+1} Z^n) \prod_{i=1}^{n} (1-\Lambda_i M_{i-1} Z^{i-1})(1-\Lambda_i M_i Z^i) \right)^{-1} \qquad (3.8)
$$

where M_0 and M_n are to be replaced by unity. The U(1) label carried by Z is the degree in the n-plet (1, 0, . . . , 0) of SU(n).

3.7. Plethysms based on (1) ⊕ (1) of SU(2)

The ingredient generating functions are $\mathscr{I}(\Lambda, M', M'', Z)$ for SU(4) ⊃ SU(2) × SU(2) × U(1) and $\mathscr{C}(M', M'', M)$, the SU(2) Clebsch–Gordan generating function. From the

SU(4) ⊃ SU(2) × SU(2) × U(1) integrity basis (Sharp 1972) we get for the SU(4) ⊃ SU(2) × SU(2) × U(1) function

$$\mathscr{J}(\Lambda, M', M'', Z) = \frac{1}{(1-\Lambda_1M'Z)(1-\Lambda_1M'')(1-\Lambda_2Z^2)(1-\Lambda_2)(1-\Lambda_3M'Z)(1-\Lambda_3M''Z^2)} \times\left(\frac{1}{1-\Lambda_2M'M''Z}+\frac{\Lambda_1\Lambda_3Z^2}{1-\Lambda_1\Lambda_3Z^2}\right). \tag{3.9}$$

The SU(2) Clebsch–Gordan generating function is, trivially,

$$\mathscr{C}(M', M'', M) = \frac{1}{(1-M'M)(1-M''M)(1-M'M'')}. \tag{3.10}$$

Substituting (3.10) into (3.9) with the help of (3.2), and multiplying by $(1-\Lambda_4Z^2)^{-1}$ we get

$$\begin{aligned}\mathscr{F} = {} & \frac{1}{(1-\Lambda_2)(1-\Lambda_2Z^2)(1-\Lambda_4Z^2)}\Bigg[\left(\frac{1}{1-\Lambda_2ZM'M''}+\frac{\Lambda_1\Lambda_3Z^2}{1-\Lambda_1\Lambda_3Z^2}\right)\\
& \times\Bigg(\frac{1}{(1-\Lambda_1ZM'M)(1-\Lambda_1M''M)(1-\Lambda_3ZM'M)(1-\Lambda_3Z^2M''M)(1-\Lambda_1\Lambda_3Z^3M'M'')}\\
& +\frac{\Lambda_1^2ZM'M''}{(1-\Lambda_1ZM'M)(1-\Lambda_1M''M)(1-\Lambda_1^2ZM'M'')(1-\Lambda_1\Lambda_3Z^3M'M'')(1-\Lambda_1\Lambda_3ZM'M'')}\\
& +\frac{\Lambda_3^2Z^3M'M''}{(1-\Lambda_3ZM'M)(1-\Lambda_3Z^2M''M)(1-\Lambda_3^2Z^3M'M'')(1-\Lambda_1\Lambda_3Z^3M'M'')(1-\Lambda_1\Lambda_3ZM'M'')}\\
& +\frac{\Lambda_1\Lambda_3ZM'M''}{(1-\Lambda_1ZM'M)(1-\Lambda_1M''M)(1-\Lambda_3ZM'M)(1-\Lambda_1\Lambda_3Z^3M'M'')(1-\Lambda_1\Lambda_3ZM'M'')}\\
& +\frac{\Lambda_1\Lambda_3^2Z^3M'M''^2M}{(1-\Lambda_1M'M)(1-\Lambda_3ZM'M)(1-\Lambda_3Z^2M''M)(1-\Lambda_1\Lambda_3Z^3M'M'')(1-\Lambda_1\Lambda_3ZM'M'')}\Bigg)\\
& +\frac{\Lambda_2ZM'M''M^2}{(1-\Lambda_1ZM'M)(1-\Lambda_1M''M)(1-\Lambda_3ZM'M)\times(1-\Lambda_3Z^2M''M)(1-\Lambda_2ZM'M'')(1-\Lambda_2ZM'M''M^2)}\Bigg];\end{aligned} \tag{3.11}$$

M in (3.11) carries the SU(2) label, Z carries the degree in the first SU(2) doublet, and M' and M'' carry the representation labels of the intermediate SU(2) × SU(2) group.

3.8. Plethysms based on (1, 0, . . . , 0)(0) ⊕ (0, 0, . . . , 0)(1) of SU(n) × SU(2)

The integrity basis for SU(n + 2) ⊃ SU(n) × SU(2) × U(1) is given implicitly by Mickelsson (1970) and explicitly by Sharp (1972). When translated into a generating function and multiplied by $(1-\Lambda_{n+2}Z^n)^{-1}$ it becomes

$$\mathscr{F}=\frac{1}{(1-\Lambda_{n+2}Z^n)\prod_{k=1}^{n+1}[(1-A_k^1)(1-A_k^3)]}\sum_{[i,j]}\frac{\Pi' A_{i'j'}}{\Pi_{[i,j]}(1-A_{ij})\,\Pi_{[k]}(1-A_k^2)} \tag{3.12}$$

where $A_k^1=\Lambda_k M_k Z^k$, $A_k^2=\Lambda_k M_{k-1}Z^{k-1}$, $A_k^3=\Lambda_k M_{k-2}Z^{k-2}$, $A_{ij}=\Lambda_i\Lambda_j M_{i-1}M_{j-1}Z^{i+j-2}$, $M_1, M_2, \ldots, M_{n-1}$ carry the SU(n) labels, M the SU(2) label, $M_0=M_n=1$, $M_{-1}=M_{n+1}=0$, Z carries the degree in the SU(n) n-plet. The sum in (3.12) is over all sets of pairs (including the null set) $[i,j]$ satisfying the following conditions: (i) for each pair (i,j) we have $1\leqslant i\leqslant j-2\leqslant n-1$; (ii) no two pairs may overlap, i.e. have $i<i'<j<j'$ or $i'<i<j'<i$; (iii) there must be no 'internal spaces', i.e., for each pair (i,j) there must be $j-i-2$ other pairs (i',j') with $i\leqslant i'$ and $j'\leqslant j$. The product $\Pi_{[k]}$ is over all k for which $1<k<n+1$ and for which there is no pair (i,j) of $[i,j]$ such that $i<k<j$; Π' is over those pairs (i,j) of the set $[i,j]$ for which either (i) there is no other pair (i',j') of $[i,j]$ for which $i'<i$ and $j<j'$ or (ii) there is at least one other pair of $[i,j]$ for which $i'<i$ and $j'=j$.

3.9. Plethysms based on (10)(00) ⊕ (00)(10) of SU(3) × SU(3)

The integrity basis for this problem is given implicitly by Mickelsson's (1970) branching rules. We have determined it by examining the SU(3) × SU(3) × U(1) content of a large number of low-lying representations of SU(6). Since we have not written it as a generating function (it would be a sum of 45 fractions) or checked it analytically, it is at present an educated guess; however we are reasonably sure it is correct.

The integrity basis contains 25 elements, $\Lambda_6 Z^3$ and 24 others, which we denote by letters, in order to refer to them more easily below. $\Lambda_1^{\lambda_1}\Lambda_2^{\lambda_2}\Lambda_3^{\lambda_3}\Lambda_4^{\lambda_4}\Lambda_5^{\lambda_5}M_1^{m_1}M_2^{m_2}M_1'^{m_1'}M_2'^{m_2'}Z^z$ is denoted by $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5;\ m_1, m_2, m_1', m_2', z)$, where m_1, m_2 and m_1', m_2' are the labels of the two SU(3) groups and z is the degree in the first SU(3) × SU(3) triplet. The remaining 24 members of the integrity basis are

$a=(10000;\ 10, 00, 1)$ $\qquad$ $a'=(10000;\ 00, 10, 0)$

$b=(01000;\ 01, 00, 2)$ $\qquad$ $c=(01000;\ 10, 10, 1)$

$b'=(01000;\ 00, 01, 0)$ $\qquad$ $d=(00100;\ 00, 00, 3)$

$e=(00100;\ 01, 10, 2)$ $\qquad$ $e^*=(00100;\ 10, 01, 1)$

$d^*=(00100;\ 00, 00, 0)$ $\qquad$ $b'^*=(00010;\ 00, 10, 3)$

$c^*=(00010;\ 01, 01, 2)$ $\qquad$ $b^*=(00010;\ 10, 00, 1)$

$a'^*=(00001;\ 00, 01, 3)$ $\qquad$ $a^*=(00001;\ 01, 00, 2)$

$f=(10100;\ 01, 01, 2)$ $\qquad$ $g=(10010;\ 00, 01, 3)$

$g'=(10010;\ 01, 00, 2)$ $\qquad$ $h=(10001;\ 00, 00, 3)$

$j=(01010;\ 10, 01, 4)$ $\qquad$ $k=(01010;\ 00, 00, 3)$

$j^*=(01010;\ 01, 10, 2)$ $\qquad$ $g'^*=(01001;\ 10, 00, 4)$

$g^*=(01001;\ 00, 10, 3)$ $\qquad$ $f^*=(00101;\ 10, 10, 4)$.

Because of syzygies the following pairs of basis elements should not appear in the same product: *cf, cg, cg', ch, c*f*, c*g*, c*g'*, c*h, eg, eg'*, ej, ek, e*g*, e*g', e*j*, e*k, ff*, fg*, fg'*, fj, fj*, fk, f*g, f*g', f*g*, f*j, f*k, gg*, gg'*, gj*, g'g*, g'g'*, g'j, g*j, g'*j*, jj**.

3.10. Two-rowed plethysms based on (10 . . . 0)(1) of SU(n) × SU(2)

The structure of this generating function is independent of n for $n \geqslant 5$, and in fact can be determined from the solution of the problem for $n = 3$. The result for $n \geqslant 5$ is

$$\mathscr{F} = \frac{1}{(1-\Lambda_1 M_1 T)(1-\Lambda_2 M_1^2)(1-\Lambda_2^2 M_2^2)(1-\Lambda_2^3 M_3^2)(1-\Lambda_1\Lambda_2 M_3 T)(1-\Lambda_2^2 M_4)}$$
$$\times\left(\frac{1+\Lambda_1^2\Lambda_2^2 M_1 M_2 M_3}{(1-\Lambda_1^2\Lambda_2 M_1 M_3)(1-\Lambda_1^2 M_2)}+\frac{\Lambda_2^2 M_1 M_3 T^2+\Lambda_2^3 M_1 M_2 M_3 T^2}{(1-\Lambda_2 M_2 T^2)(1-\Lambda_2^2 M_1 M_3 T^2)}\right.$$
$$\left.+\frac{\Lambda_2 M_2 T^2+\Lambda_1\Lambda_2 M_1 M_2 T+\Lambda_1\Lambda_2^2 M_2 M_3 T+\Lambda_1^2\Lambda_2^3 M_1 M_2^2 M_3 T^2}{(1-\Lambda_2 M_2 T^2)(1-\Lambda_1^2 M_2)}\right). \quad (3.13)$$

M_1, M_2, M_3, M_4 carry the first four SU(n) labels, T the SU(2) label. For $n = 4$, put $M_4 = 1$; for $n = 3$, put $M_4 = 0$, $M_3 = 1$; for $n = 2$, put $M_4 = M_3 = 0$, $M_2 = 1$.

We determined $\mathscr{F}$ by examining the SU(3)×SU(2) content of low two-rowed representations of SU(6). It was checked by converting it to a generating function for SU(3)×SU(2) weights and comparing numerically with the corresponding generating function for SU(3)×U(1) weights in two-rowed representations of SU(6), obtained from the SU(6)⊃SU(3)×SU(3)×U(1) generating function corresponding to the integrity basis found in subsection (3.9).

The chain SU(6)⊃SU(3)×SU(2) is of physical interest in connection with the quark model (So and Strottman 1979). For two-rowed representations of SU(6), the generating function (3.13), with $M_4 = 0$, $M_3 = 1$, defines a complete set of states in the SU(3)×SU(2) basis; Λ_1 and Λ_2 carry the first two SU(6) representation labels.

4. Plethysms for finite groups

The generating function for plethysms based on a representation Γ of a finite group G is a particular case of a generating function $\mathscr{B}$ for reduction of representations of a continuous group G′ to representations of its finite subgroup G. In our problem G′ = U(n), where n is the dimension of Γ. An *ab initio* method of doing that would start with the generating function

$$\mathscr{f}'(\Lambda, W) = \sum_{\lambda} \Lambda^{\lambda} \sum_{w} m_{\lambda,w} W^{w} \quad (4.1)$$

for characters of representations of G′ (Patera and Sharp 1979b). In (4.1) the exponents of Λ fix an irreducible representation of G′ and $m_{\lambda,w}$ is the multiplicity of a weight w in the representation (λ). One substitutes for the variables of the character the numerical values corresponding to each class s of elements of G, thereby converting $\mathscr{f}'(\Lambda, W)$ into a generating function

$$\mathscr{f}_s(\Lambda) = \sum_{\lambda} \Lambda^{\lambda} \chi_{s,\lambda} \quad (4.2)$$

for characters $\chi_{s,\lambda}$ of the (reducible) representation of the subgroup G contained in (λ).

Multiplying (4.2) by the complex conjugate character $\chi^*_{s,i}$ of the irreducible representation Γ_i of G, and summing over s, one gets the desired generating function

$$\mathscr{B}_i(\Lambda) = \sum_\lambda n_{\lambda,i}\Lambda^\lambda \tag{4.3}$$

where $n_{\lambda,i}$ is the multiplicity of Γ_i in (λ).

Again the general procedure can often be shortened using the artifices of § 2. For plethysms based on representations of point groups considered below one can always use one of the chains

$$\begin{aligned} &\mathrm{U}(n) \supset \mathrm{O}(n) \supset \mathrm{O}(3) \supset \mathrm{G} \qquad (n \text{ odd}) \\ &\mathrm{U}(n) \supset \mathrm{Sp}(n) \supset \mathrm{SU}(2) \supset \mathrm{G} \qquad (n \text{ even}) \end{aligned} \tag{4.4}$$

and profit from the known generating functions for $\mathrm{O}(3) \supset \mathrm{G}$ and $\mathrm{SU}(2) \supset \mathrm{G}$ (Patera *et al* 1978, Desmier and Sharp 1979) and substitute them according to (3.2) in the generating function $\mathrm{U}(n) \supset \mathrm{O}(3)$ or $\mathrm{U}(n) \supset \mathrm{SU}(2)$.

Plethysm generating functions based on two-dimensional representations of a point group G differ only by a factor $(1-\Lambda_2)^{-1}$ from the generating functions for $\mathrm{SU}(2) \supset \mathrm{G}$.

For plethysms based on the vector representation of G one substitutes the generating function for $\mathrm{O}(3) \supset \mathrm{G}$ into the generating function (3.4) for $\mathrm{U}(3) \supset \mathrm{O}(3)$ replacing M^2 by L with the result

$$\mathscr{B} = \frac{1}{(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3)(\Lambda_1-\Lambda_2)}[\Lambda_1(1+\Lambda_2)(1-\Lambda_1^2L^2)B_i(\Lambda_1L) - \{\Lambda_1 \leftrightarrow \Lambda_2\}] \tag{4.5}$$

where $(1-L^2)B_i(L)$ is the generating function for $\mathrm{O}(3) \supset \mathrm{G}$ and $\{\Lambda_1 \leftrightarrow \Lambda_2\}$ denotes a term with Λ_1 and Λ_2 interchanged; exponents of L are the angular momenta labelling O(3) representations.

Below we list the generating functions $\mathscr{B}_i$, $i = 1, 2, 3, 4, 5$, for plethysms based on the three-dimensional irreducible representation Γ_5 of the octahedral group O (Γ_1 is the identity representation; Γ_2 is the pseudoscalar representation; Γ_3 is the two-dimensional one; Γ_4 and Γ_5 are of dimension three; Γ_5 is the defining representation):

$$\begin{aligned} \mathscr{B}_1 = {} & \frac{1}{(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3)(1-\Lambda_1^3L^3)(1-\Lambda_2^4L^4)} \\ & \times\left(\frac{L^4}{1-\Lambda_1^4L^4}(\Lambda_1\Lambda_2^4+\Lambda_1^2\Lambda_2^3+\Lambda_1^3\Lambda_2^2+\Lambda_1^4\Lambda_2+\Lambda_1\Lambda_2^3+\Lambda_1^2\Lambda_2^2+\Lambda_1^3\Lambda_2+\Lambda_1^4)\right. \\ & \left.+\frac{1}{1-\Lambda_2^3L^3}[1+L^3(\Lambda_1\Lambda_2^3+\Lambda_1^2\Lambda_2^2+\Lambda_1^3\Lambda_2+\Lambda_1\Lambda_2^2+\Lambda_1^2\Lambda_2)]\right) \end{aligned} \tag{4.6}$$

$$\begin{aligned} \mathscr{B}_2 = {} & \frac{L^6}{(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3)(1-\Lambda_1^3L^3)(1-\Lambda_2^4L^4)} \\ & \times\left(\frac{1}{1-\Lambda_1^4L^4}(\Lambda_1^3\Lambda_2^3+\Lambda_1^4\Lambda_2^2+\Lambda_1^5\Lambda_2+\Lambda_1^6+\Lambda_1^4\Lambda_2^3+\Lambda_1^5\Lambda_2^2+\Lambda_1^6\Lambda_2+\Lambda_1^7\Lambda_2^4L^4)\right. \\ & \left.+\frac{L}{1-\Lambda_2^3L^3}(\Lambda_2^6+\Lambda_1\Lambda_2^5+\Lambda_1^2\Lambda_2^4+\Lambda_1\Lambda_2^6+\Lambda_1^2\Lambda_2^5+\Lambda_1^3\Lambda_2^4)\right) \end{aligned} \tag{4.7}$$

$$\mathscr{B}_3=\frac{1}{(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3)(1-\Lambda_1^3L^3)(1-\Lambda_2^4L^4)}$$
$$\times\Big(\frac{1}{1-\Lambda_1^4L^4}[L^4(\Lambda_1^3\Lambda_2+\Lambda_1^4\Lambda_2+\Lambda_1^4)+L^6(\Lambda_1^3\Lambda_2^3+\Lambda_1^4\Lambda_2^2+\Lambda_1^5\Lambda_2+\Lambda_1^6+\Lambda_1^3\Lambda_2^4$$
$$+\Lambda_1^4\Lambda_2^3+\Lambda_1^5\Lambda_2^2+\Lambda_1^6\Lambda_2)+L^8(\Lambda_1^5\Lambda_2^3+\Lambda_1^6\Lambda_2^2+\Lambda_1^5\Lambda_2^4+\Lambda_1^6\Lambda_2^3+\Lambda_1^7\Lambda_2^2)]$$
$$+\frac{1}{1-\Lambda_2^3L^3}[L^2(\Lambda_2^2+\Lambda_1\Lambda_2+\Lambda_1^2)$$
$$+L^4(\Lambda_2^4+\Lambda_1\Lambda_2^3+\Lambda_1^2\Lambda_2^2+\Lambda_1\Lambda_2^4+\Lambda_1^2\Lambda_2^3+\Lambda_1^3\Lambda_2^2)+\Lambda_1^3\Lambda_2^3L^5]\Big) \tag{4.8}$$

$$\mathscr{B}_4=\frac{1}{(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3)(1-\Lambda_1^3L^3)(1-\Lambda_2^4L^4)}$$
$$\times\Big(\frac{1}{1-\Lambda_1^4L^4}[L^3\Lambda_1^3+L^4(\Lambda_1^4+\Lambda_1^3\Lambda_2+\Lambda_1^4\Lambda_2)$$
$$+L^5(\Lambda_1^5+\Lambda_1^4\Lambda_2+\Lambda_1^3\Lambda_2^2+\Lambda_1^2\Lambda_2^3+\Lambda_1^5\Lambda_2+\Lambda_1^4\Lambda_2^2+\Lambda_1^3\Lambda_2^3+\Lambda_1^2\Lambda_2^4)$$
$$+L^7(\Lambda_1^6\Lambda_2+\Lambda_1^5\Lambda_2^2+\Lambda_1^4\Lambda_2^3+\Lambda_1^7\Lambda_2+\Lambda_1^6\Lambda_2^2+\Lambda_1^5\Lambda_2^3+\Lambda_1^4\Lambda_2^4)$$
$$+L^8(\Lambda_1^6\Lambda_2^2+\Lambda_1^5\Lambda_2^3+\Lambda_1^7\Lambda_2^2+\Lambda_1^6\Lambda_2^3+\Lambda_1^5\Lambda_2^4)]$$
$$+\frac{1}{1-\Lambda_2^3L^3}[L^3(\Lambda_1^2\Lambda_2+\Lambda_1\Lambda_2^2+\Lambda_2^3+\Lambda_1^3\Lambda_2+\Lambda_1^2\Lambda_2^2+\Lambda_1\Lambda_2^3)$$
$$+L^4(\Lambda_1^2\Lambda_2^2+\Lambda_1\Lambda_2^3+\Lambda_2^4+\Lambda_1^3\Lambda_2^2+\Lambda_1^2\Lambda_2^3+\Lambda_1\Lambda_2^4)$$
$$+L^5(\Lambda_1\Lambda_2^4+\Lambda_2^5+\Lambda_1\Lambda_2^5)+L^8(\Lambda_1^2\Lambda_2^6+\Lambda_1^3\Lambda_2^6+\Lambda_1^2\Lambda_2^7)]\Big) \tag{4.9}$$

$$\mathscr{B}_5=\frac{1}{(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3)(1-\Lambda_1^3L^3)(1-\Lambda_2^4L^4)}$$
$$\times\Big(\frac{1}{1-\Lambda_2^3L^3}[(\Lambda_1+\Lambda_2+\Lambda_1\Lambda_2)L+(\Lambda_1^2+\Lambda_1\Lambda_2+\Lambda_2^2+\Lambda_1\Lambda_2^2+\Lambda_1^2\Lambda_2)L^2$$
$$+(\Lambda_1^2\Lambda_2+\Lambda_1\Lambda_2^2+\Lambda_2^3+\Lambda_1^3\Lambda_2+\Lambda_1^2\Lambda_2^2+\Lambda_1\Lambda_2^3)L^3$$
$$+(\Lambda_1^3\Lambda_2^2+\Lambda_1^2\Lambda_2^3)L^4+\Lambda_1^3\Lambda_2^3L^5]$$
$$+\frac{1}{1-\Lambda_1^4L^4}[\Lambda_1^3L^3+(\Lambda_1^5+\Lambda_1^4\Lambda_2+\Lambda_1^3\Lambda_2^2+\Lambda_1^2\Lambda_2^3$$
$$+\Lambda_1^5\Lambda_2+\Lambda_1^4\Lambda_2^2+\Lambda_1^3\Lambda_2^3+\Lambda_1^2\Lambda_2^4)L^5$$
$$+(\Lambda_1^6+\Lambda_1^5\Lambda_2+\Lambda_1^4\Lambda_2^2+\Lambda_1^3\Lambda_2^3+\Lambda_1^6\Lambda_2+\Lambda_1^5\Lambda_2^2+\Lambda_1^4\Lambda_2^3+\Lambda_1^3\Lambda_2^4)L^6$$
$$+(\Lambda_1^6\Lambda_2+\Lambda_1^5\Lambda_2^2+\Lambda_1^4\Lambda_2^3+\Lambda_1^7\Lambda_2+\Lambda_1^6\Lambda_2^2+\Lambda_1^5\Lambda_2^3+\Lambda_1^4\Lambda_2^4)L^7]\Big). \tag{4.10}$$

The generating functions based on the three dimensional irreducible representation of the tetrahedral group T are easily obtained from (4.6)–(4.10) due to the fact that $T\subset O$. Without going into details, let us point out that the generating functions $\mathscr{D}_i$,

$j = 1, 2, 3, 4$, (the representations of T are: Γ_1 the identity, Γ_2 and Γ_3 the two mutually conjugate ones of dimension one, and Γ_4 is the representation of dimension three) for plethysms based on Γ_4 can be written in terms of generating functions for the octahedral group as follows:

$$\mathscr{D}_1 = \mathscr{B}_1 + \mathscr{B}_2 \qquad \mathscr{D}_2 = \mathscr{D}_3 = \mathscr{B}_3 \qquad \mathscr{D}_4 = \mathscr{B}_4 + \mathscr{B}_5. \tag{4.11}$$

5. Integrity bases

Each generating function given in §§ 3 and 4 describes a set of tensors. The set of tensors in each case can also be described in terms of a finite integrity basis, a finite set of elementary tensors or multiplets (the elementary permissible diagrams of Devi and Moshinsky (1969)). All the tensors of the set are then stretched (representation labels additive) products of powers of the elementary ones. Because of syzygies (relations between the elementary multiplets) certain products are incompatible and should be discarded to avoid double counting.

A generating function is a convenient way of representing an integrity basis mathematically and makes possible various manipulations on them such as converting an integrity basis for group tensors into the corresponding one for subgroup tensors or the reverse.

Since the generating function and the integrity basis carry the same information, it should be possible to translate one into the other; that is true, but the conversion is not effortless.

The denominator factors of the generating function have the form $1-X$; the X and the numerator terms Y are products of powers of dummy variables carrying group labels and other information, such as degrees, as exponents. The elementary tensors correspond to the X and certain of the Y (the other Y are products of powers of the elementary tensors). Once the elementary X and Y are identified, it is straightforward, by standard methods, to determine the algebraic form of the tensors they denote. The absence of products of elementary tensors implies corresponding syzygies concerning them.

The simplest way to construct a tensor of a plethysm corresponding to a Young tableau $\{\lambda\}$ is to construct the states of the representation (λ) of U(n). They will in the first instance be of degree $\lambda_1, \lambda_2, \ldots, \lambda_n$ in polynomials which, respectively, are antisymmetric in the first $1, 2, \ldots, n$ copies of the basic n-component Γ tensor. To get a tensor of the desired symmetry, linear in each of $p = \Sigma_i i\lambda_i$ copies of the tensor, it is necessary to apply the Young symmetry operator to the states already obtained. By permuting $1, 2, \ldots, p$ one obtains $D_{\{\lambda\}}$ linearly independent copies of the tensor.

We conclude this section by giving the finite integrity basis corresponding to plethysms based on (01) of O(5) and (1)⊕(1) of SU(2)×SU(2). Bases for most of the others are found in the literature, or are easily constructed from the generating functions.

For plethysms based on (01) of O(5) there are 17 members of the finite integrity basis. One of them has $\lambda_5 = 1$ and all other labels zero. The rest, in the notation $(\lambda_1\lambda_2\lambda_3\lambda_4, \mu_1\mu_2)$ are (1001, 20), (1010, 20), (0101, 20), (0200, 02), (0020, 02), (1100, 01), (0011, 01), (0110, 01), (0110, 21), (1110, 20), (0111, 20), (1101, 20), (1011, 20), (1020, 21), (0201, 21), (1111, 20).

Because of syzygies the following products should be discarded: the square of any one or product of any two of (0110, 21), (1110, 20), (0111, 20), (1101, 20), (1011, 20), (1020, 21), (0201, 21), (111, 20); the product of any of (1101, 20), (1011, 20), (1111, 20) with any of (0200, 02), (0020, 02), (1100, 01), (0011, 01), (0110, 01); (0111, 20) or (0201, 21) with any of (1010, 20), (0020, 02), (0011, 01), (0110, 01); (1110, 20) or (1020, 21) with (0101, 20), (0200, 02), (1100, 01) or (0110, 01); (1001, 20) with any of (0200, 02), (0020, 02), (1100, 01), (0011, 01), (0110, 01), (0110, 21), (1020, 21), (0201, 21); (1010, 20) with (0101, 20), (0200, 02), (1100, 01), (0110, 21) or (1101, 20); (0101, 20) with (0020, 02), (0011, 01), (0110, 21), (1011, 20); the square of (1100, 01), (0011, 01), (0110, 01).

For plethysms based on (1)(0)⊕(0)(1) of SU(2)×SU(2), there are 14 members of the finite integrity basis. One has $\lambda_4 = 1$, $z = 2$. The others, in the notation $(\lambda_1\lambda_2\lambda_3; zm'm'', m)$, are (010; 000, 0), (010; 200, 0), (010; 111, 0), (101; 200, 0), (100; 110, 1), (100; 001, 1), (001; 110, 1), (001; 201, 1), (101; 311, 0), (200; 111, 0), (101; 111, 0), (002; 311, 0), (010; 111, 2).

The following pairs are incompatible: (100; 110, 1) with (002; 311, 0); (001; 201, 1) with (002; 311, 0); (001; 010, 1) with (200; 111, 0); (001; 201, 1) with (200; 111, 0); (200; 111, 0) with (002; 311, 0); (010; 111, 2) with any of (101; 200, 0), (101; 311, 0), (101; 111, 0), (200; 111, 0) or (002; 311, 0); and (010; 111, 0) with (101; 200, 0). In addition, the following product of three elementary multiplets is incompatible: (100; 110, 1), (001; 201, 1), (101; 111, 0).

6. Subjoining of a semi-simple group to a semi-simple group

Subjoining is a weaker relation between two algebras than an embedding; thus if A ⊃ B then B is subjoined to A; however the inverse is not always true. The present section is devoted to this relation which, as far as we know, has not been described previously. It proves to be quite advantageous in calculating some generating functions.

Consider two semi-simple Lie algebras A and B of ranks $l_A \geq l_B$; denote by ϕ and ψ respectively representations of A and B of finite dimensions, and by W^ϕ and W^ψ the corresponding weight systems. If one has also A ⊃ B then ϕ may be decomposed:

$$\phi \supset \psi = \bigoplus_i \psi_i \tag{6.1}$$

and

$$\mathscr{P} W^\phi = W^\psi = \sum_i W^{\psi_i} \tag{6.2}$$

where ψ_i are irreducible representations of B contained in ϕ of A; $\mathscr{P}$ is a real $l_A \times l_B$ matrix projecting the weights of W^ϕ into weights of W^ψ.

We say that B is *subjoined* to A if for *all* finite representations ϕ of A there exists a real $l_A \times l_B$ matrix $\mathscr{P}$ such that

$$\mathscr{P} W^\phi = \sum_i W^{\psi_i} - \sum_j W^{\psi_j} \tag{6.3}$$

where the summations extend over some irreducible representations of B. We write A > B.

The projection matrix $\mathscr{P}$ specifies the way B is subjoined to A; $\mathscr{P}$ is not uniquely determined by (6.3) but it is not necessary to distinguish two projection matrices which lead to the same sums on the right side of (6.3). It is natural to specify the subjoining of B to A, i.e. to fix $\mathscr{P}$, using the lowest representation of A. The defining property (6.3) is given in terms of weights of representations; therefore one can also speak about a semi-simple Lie group subjoined to another semi-simple group. For simplicity we write also $\phi \supset \oplus_i \psi_i \ominus_j \psi_j$ instead of (6.3).

The simplest example which is not a group–subgroup relation is $SU(2) > O(3)$. It can be specified using as ϕ the representation of dimension two. The weight system consists of two elements, $W^{\phi} = [1, -1]$. Then $\mathcal{P}$ is a 1×1 matrix, $\mathcal{P} = 2$, and

$$\mathcal{P}W^{\phi} = 2[1, -1] = [2, -2] = [2, 0, -2] - [0] \tag{6.4}$$

where $[2, 0, -2]$ and $[0]$ are the weight systems of O(3) representations of dimension three and one. It is obvious that such a 'reduction' is possible and unique for any representation of SU(2) using the same $\mathcal{P} = 2$. Furthermore we can write a generating function for $SU(2) > O(3)$. One easily verifies that it is

$$\mathcal{F}(A, B) = \frac{1}{(1+A)(1-AB^2)} = 1 + A(B^2 - 1) + A^2(B^4 - B^2 + 1) + \ldots. \tag{6.5}$$

Here the negative signs in the power series are in one-to-one correspondence with the negative terms in (6.3).

A less trivial example is the case $O(5) > SU(2) \times SU(2)$ where the subjoining is specified by

$$\mathcal{P}W^{(1,0)} = W^{(1)(1)}, \tag{6.6}$$

where (10) and (1) denote respectively the four- and two-dimensional representations of O(5) and SU(2). Here we are dealing with $(1, 0) > (1)(1)$ which differs from the usual embedding of $SU(2) \times SU(2)$ in O(5) which is given by $(1, 0) \supset (1)(0) \oplus (0)(1)$. The generating function for $O(5) > SU(2) \times SU(2)$ which follows from (6.6) is given by

$$\mathcal{F} = \frac{1}{(1-N_1^2)(1-N_2M_1^2)(1-N_2M_2^2)} \left(\frac{1}{1-N_1M_1M_2} - \frac{N_2}{1+N_2} \right) \tag{6.7}$$

where N_1, N_2 carry the O(5) labels, M_1, M_2 those of $SU(2) \times SU(2)$. To verify (6.7), convert it into a generating function for $SU(2) \times SU(2)$ weights and compare it to the known generating function for O(5) weights.

An application of (6.7) would be the derivation of equation (3.6) for $U(4) \supset SU(2) \times SU(2)$ using the chain $SU(4) \supset O(5) > SU(2) \times SU(2)$.

A still more complicated example is the case $SU(4) > O(5)$. The subjoining here is specified by

$$\mathcal{P}W^{(1,0,0)} = W^{(0,1)} - W^{(0,0)} \tag{6.8}$$

where $(1, 0, 0)$, $(0, 1)$ and $(0, 0)$ denote respectively the representation of SU(4) of dimension four and O(5) of dimension five and one. The generating function for $SU(4) > O(5)$ implied by (6.8) provides an easy way to reduce any higher representation of SU(4) to representations of the subjoint group O(5); it is

$$\mathcal{F} = \frac{1}{(1+N_1)(1+N_3)(1-N_2)(1-N_1M_2)(1-N_3M_2)(1-N_2M_1^2)} \times \left(\frac{1}{1-N_1N_3M_1^2} - \frac{N_2M_2}{1+N_2M_2} \right) \tag{6.9}$$

where N_1, N_2, N_3 carry the representation labels of SU(4), M_1, M_2 those of O(5). The formula (6.9) can be verified by converting it into an SU(4) ⊃ SU(2) × SU(2) generating function. This is done by substituting into it, using (3.2), the known generating function

$$\frac{1}{(1-M_1S)(1-M_1T)(1-M_2)(1-M_2ST)}$$

for O(5) ⊃ SU(2) × SU(2) branching rules. The resulting SU(4) ⊃ SU(2) × SU(2) function can be compared with the known correct formula (see, for example, equation (3.6)).

An obvious application of equation (6.9) is to substitute it, with (3.2), into the known generating function for SU(5) ⊃ SU(4),

$$\frac{1}{(1-\Lambda_1)(1-\Lambda_1N_1)(1-\Lambda_2N_1)(1-\Lambda_2N_2)(1-\Lambda_3N_2)(1-\Lambda_3N_3)(1-\Lambda_4N_3)(1-\Lambda_4)(1-\Lambda_5)}$$

to obtain equation (3.7) for the branching rules U(5) ⊃ O(5).

As another example we mention the subjoining of G_2 to Sp(6) so that the defining representation (100) of Sp(6) contains the representations (10)⊖(00) of G_2, i.e., a septet minus a scalar. Since the branching rules for the chain SU(7) ⊃ SU(6) ⊃ Sp(6) are well known, the subjoining of G_2 to Sp(6) would be helpful in obtaining branching rules for SU(7) ⊃ G_2 and hence the generating function for G_2 plethysms based on a (10) tensor. In this case we could also work with the chain SU(7) ⊃ O(7) ⊃ G_2.

We have not found any relation between the subjoint and parent groups other than the implied relation between their weight diagrams.

7. Antisymmetric and symmetric plethysms of SU(2)

The generating functions of preceding sections were made for computation of general plethysms based on a fixed representation Γ of a group G. In this section we derive a novel type of generating function for plethysms. We restrict the type of plethysm we consider (to antisymmetric or symmetric of fixed degree), but allow Γ to run through all irreducible representations of SU(2). These plethysms can again be described by a single generating function. In principle, the procedure may be generalised to other groups and other plethysms.

We denote by $\Phi_p(L, S)$ the generating function for completely antisymmetric plethysms of degree p based on an SU(2) tensor of an arbitrary rank l (dimension $2l+1$). When $\Phi_p(L, S)$ is expanded into a power series

$$\Phi_p(L, S) = \sum_{l,s} n_{pls} L^l S^s \tag{7.1}$$

the variable L carries the rank l as its exponent, and S carries the SU(2) representation label s; n_{pls} is the multiplicity of an SU(2) representation of dimension $2s+1$ contained in the antisymmetric plethysm of degree p based on a tensor of rank l. Our problem is to find $\Phi_p(L, S)$ as an explicit function of the variables $L^{1/2}$ and $S^{1/2}$. Our strategy is first to write a recursion relation for multiplicities N_{plm} of weights in these plethysms, then rewrite it as a recurrence relation for the multiplicities n_{pls} of representations, and finally to transform that into a recursion relation for $\Phi_p(L, S)$.

We begin with a recurrence formula for multiplicities of weights,

$$N_{plm} = N_{p\,l-1\,m} + N_{p-1\,l-1\,m-l} + N_{p-1\,l-1\,m+l} + N_{p-2\,l-1\,m} \qquad l \geq 1, \tag{7.2}$$

which expresses the fact that each weight of degree p can be formed from weights of an $(l-1)$-tensor together with (a) neither weight l or $-l$, (b) one weight l, (c) one weight $-l$, (d) both weights l and $-l$. Because of antisymmetry a particular weight can be used at most once. The multiplicity n_{pls} of tensors of rank s can be expressed in terms of the multiplicities of weights:

$$n_{pls} = N_{pls} - N_{pls+1}. \qquad (7.3)$$

Substitution for N_{pls} and N_{pls+1} from (7.2) leads to the corresponding recurrence formula for n_{pls}:

$$\begin{aligned} n_{pls} = {} & n_{p\,l-1\,s} + n_{p-1\,l-1\,s-l} + n_{p-1\,l-1\,s+l} - n_{p-1\,l-1\,l-s-1} \\ & + n_{p-2\,l-1\,s} \qquad l \geqslant 1. \end{aligned} \qquad (7.4)$$

It is understood that $n_{pls} = 0$ if any subscript is negative; in deriving (7.4) the relation $N_{pl-m} = N_{plm}$ was used. Multiplying (7.4) by $L^l S^s$ and summing over integer and half-odd values of l and s, one is led to a recurrence relation for the generating function $\Phi_p(L, S) = \Sigma_{l,s} n_{pls} L^l S^s$:

$$\begin{aligned} \Phi_p(L, S) = \frac{1}{1-L}\Big(& LS\Phi_{p-1}(LS, S) + \frac{L}{S}\Phi_{p-1}\Big(\frac{L}{S}, S\Big) - L\Phi_{p-1}\Big(LS, \frac{1}{S}\Big) \\ & + L\Phi_{p-2}(L, S) + \delta_{p0}(1 + L^{1/2}) + \delta_{p1}(1 + L^{1/2}S^{1/2}) + \delta_{p2}L^{1/2}\Big). \end{aligned} \qquad (7.5)$$

In (7.5) it is understood that terms of negative degree in S in the expansion of the right side are to be discarded. A prescription for discarding the negative power part of a rational function $f(\eta)$ is

$$f(\eta) \to \Sigma \operatorname{Res} f(\eta')/(\eta' - \eta) \qquad (7.6a)$$

or

$$f(\eta) = \Sigma \operatorname{Res} f(\eta'^{-1})/\eta'(1 - \eta'\eta). \qquad (7.6b)$$

The right side of (7.6) is the sum of residues of poles in η' within the unit circle ($|\eta| < 1$ for this purpose).

To obtain Φ_p from Φ_{p-1} with the help of (7.5) is a straightforward exercise in residues. We find

$$\begin{aligned} \Phi_0(L, S) &= (1 - L^{1/2})^{-1} \\ \Phi_1(L, S) &= (1 - L^{1/2}S^{1/2})^{-1} \\ \Phi_2(L, S) &= L^{1/2}[(1-L)(1-L^{1/2}S)]^{-1} \\ \Phi_3(L, S) &= (L + L^{5/2}S^{3/2})[(1-L^2)(1-LS)(1-L^{1/2}S^{3/2})]^{-1} \\ \Phi_4(L, S) &= (L^{3/2} + L^3S^3)[(1-L)(1-L^{3/2})(1-LS^2)(1-L^{1/2}S^2)]^{-1}. \end{aligned} \qquad (7.7)$$

With (7.4) as the starting point, a recurrence formula in l can be deduced for

$$F_l(U, S) = \sum_{p,s} n_{pls} U^p S^s,$$

where n_{pls} has the same meaning as in (7.1).

In fact,

$$F_l(U,S)=(1+US^l)(1+US^{-l})F_{l-1}(U,S)-(U,S^{l-1})F_{l-1}\left(U,\frac{1}{S}\right) \qquad l\geqslant 1; \qquad (7.8)$$

terms of negative degree in S on the right side of (7.8) should be discarded. $F_l(U,S)$ is a polynomial of degree $2l+1$ in U and $\frac{1}{2}l(l+1)$ or $\frac{1}{2}(l+\frac{1}{2})^2$ in S according to whether l is integral or half-odd.

Symmetric plethysms based on SU(2) tensors are also of considerable interest. They correspond to the polynomial tensors of Patera and Sharp (1979a). By methods similar to those leading to (7.4), we find a recurrence relation for n'_{pls}, the number of s tensors in the symmetric plethysm of degree p based on an l tensor:

$$n'_{pls}=\sum_{u=0}^{p}\sum_{v=0}^{u} n'_{p-u,l-1,s-(u-2v)l}-\sum_{u=0}^{p}\sum_{v=0}^{u} n'_{p-u,l-1,(u-2v)l-s-1} \qquad l\geqslant 1. \qquad (7.9)$$

$n'_{pls}=0$ if any subscript is negative.

Multiplying (7.9) by L^lS^s and summing over integer and half-odd values of l and s we obtain a recurrence formula for the generating function $\Phi'_p(L,S)=\Sigma_{l,s}\, n'_{pls}L^lS^s$:

$$\Phi'_p(L,S)=\frac{1}{1-L}\Bigg(\sum_{u=1}^{p}\sum_{v=0}^{u} LS^{u-2v}\Phi'_{p-u}(LS^{u-2v},S)$$
$$-\sum_{u=1}^{p}\sum_{v=0}^{[\frac{1}{2}u]} LS^{u-2v-1}\Phi'_{p-u}(LS^{u-2v},S^{-1})+1+L^{1/2}S^{\frac{p}{2}}\Bigg). \qquad (7.10)$$

Terms of negative degree in S on the right of (7.10) are to be discarded, using (7.6). Iterating (7.10), we find

$$\begin{aligned}
\Phi'_0(L,S)&=(1-L)^{-1/2}\\
\Phi'_1(L,S)&=(1-L^{1/2}S^{1/2})^{-1/2}\\
\Phi'_2(L,S)&=[(1-L)(1-L^{1/2}S)]^{-1} \qquad\qquad (7.11)\\
\Phi'_3(L,S)&=(1+L^{3/2}S^{3/2})[(1-L^2)(1-LS)(1-L^{1/2}S^{3/2})]^{-1}\\
\Phi'_4(L,S)&=(1+L^{3/2}S^3)[(1-L)(1-L^{3/2})(1-LS^2)(1-L^{1/2}S^2)]^{-1}.
\end{aligned}$$

Parenthetically, we note that a generating function for third-degree plethysms of mixed symmetry is easily obtained. Write $\Phi''_{3ls}=\Sigma_{ls}n''_{3ls}L^lS^s$, where n''_{3ls} is the number of s tensors in the tensor product of three l tensors. Using the SU(2) Clebsch–Gordan generating function it is straightforward to show that

$$\Phi''_{3ls}(LS)=(1+L^{1/2}S^{1/2}+LS)[(1-L)(1-L^{1/2}S^{1/2})(1-L^{1/2}S^{3/2})]^{-1}.$$

Now substract from Φ''_{3ls} the sum of Φ_{3ls} and Φ'_{3ls}, the generating functions for antisymmetric and symmetric plethysms of degree three (equations (7.7) and (7.11)) and divide by two. We get the desired generating function for third-degree plethysms of mixed symmetry:

$$\Phi'''_3(L,S)=L^{1/2}S^{1/2}[(1-L)(1-L^{1/2}S^{1/2})(1-L^{1/2}S^{3/2})]^{-1}. \qquad (7.12)$$

A recurrence formula in l for symmetric plethysms can be deduced from (7.9). Define $F'_l(U, S) = \Sigma_{p,s} n'_{pls} U^p S^s$. Then, after some manipulation, one gets

$$F'_l(U, S) = [(1 - US^l)(1 - US^{-l})]^{-1}[F'_{l-1}(U, S) - S^{-1}F'_{l-1}(U, S^{-1})]. \tag{7.13}$$

Only non-negative powers of S are to be retained on the right. Negative powers are most simply discarded by using (7.6*a*) on the first term and (7.6*b*) on the second term. We hope to report soon on the iteration of (7.13) up to $l = 5$.

8. Concluding remarks

Generating functions for plethysms based on any representation Γ of a finite or compact continuous Lie group can, in principle, be calculated using the present procedure. Practically, however, one cannot go much higher with the dimension of Γ than $\dim \Gamma = 4$ or 5 with only hand computations. A skilful use of algebraic computer languages can, of course, put the limit much higher. At present it appears that all particular cases of plethysms of interest in applications can be treated. Our method complements the standard procedure for computing plethysms based on Littlewood's recurrence rules (King 1974), in which the dimension of Γ is unrestricted, but the degree of the plethysm is a small integer.

All the generating functions calculated recently are brought to a special form such that all numerators contain only positive terms and every factor in any denominator is such that, if expanded into a power series, it contains but positive terms. Only from a generating function of such a type can one deduce information about the existence and degrees of elements of integrity bases (cf Gaskell *et al* 1978, Patera *et al* 1978, Desmier and Sharp 1979). It is not difficult to see that the special form is not unique; consequently the choice of an integrity basis is also not unique. It is a matter of convenience in each case which choice one wants to make (Patera and Sharp 1979a).

A detailed investigation of known generating functions reveals a number of symmetry properties, not all of them being of trivial nature. Thus for instance, the generating function (3.6) for $SU(4) \supset SU(2) \times SU(2)$ does not change if Λ_1 and Λ_3 are interchanged. It reflects the symmetry under complex conjugation of the $SU(n)$ representation. In the numerator of a generating function which has been brought to a common denominator there is a correspondence between lower powers and higher powers.

A generating function in its standard form may contain several fractions. Each of the denominators contains $\frac{1}{2}(n^2 + n + l - r)$ factors, where n is the dimension of Γ on which the plethysms are based, l is the rank of G, and r is the number of its parameters. For a finite G, both l and r are zero.

It was demonstrated that for particular plethysms (the antisymmetric and symmetric ones in the example of § 7) a different type of generating function can be found which puts no restriction whatsoever on the dimension of Γ. The examples (7.7) and (7.11) imply that

$$\phi_p(L, S) = L^{(p-1)/2}\phi'_p(L, S) \qquad 0 \leq p \leq 4. \tag{8.1}$$

It would be interesting to know whether (8.1) holds for $p > 4$.

Composition laws governing groups and their representations are reflected in manipulation (composition) rules for generating functions (cf § 3).

It should be interesting to study further the subjoining of Lie algebras and the possible interpretation of that relation in physics.

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